Scope International Journal of Science, Humanities, Management and Technology. ISSN : 2455-068X
Vol.4 Issue 3 (2018) 157 - 165. Submitted 15/08/2018. Published 07/09/2018

STRONGLY CONTINUOUS COSINE FAMILY OF A DIFFERENTIAL EQUATION

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ABSTRACT

In this thesis, we relate the notion of strongly continuous cosine families of linear operators in Banach space. And the existence of solutions of the semilinear second order differential initial value problem is proved. We establish the similar results for cosine function for operators. This complements gives the representation of certain second order differential operators generate the cosine families. In particular, this theory is discussed about oscillation criteria and Leighton's oscillation theorems.

Key words: Cosine families, Banach space, oscillation criteria, Leighton's oscillation theorem.

INTRODUCTION

Many differential equations we find in nature are second order equations. In this help sheet we will demonstrate how the simplest form of the second order differential can be solved, before outlining the standard procedure for solving such an equation.

It is easy to see why second order differential equations appear in engineering and physics as the questions are often about how objects move when forces act.

As force directly relates to acceleration and acceleration is the second derivative of displacement, a second order differential equation is needed to relate force and position. It is surprising to find that these equations have properties which make them useful in a far wider range of disciplines.

As integration is the opposite of differentiation, it would seem reasonable that we would have to integrate twice to solve this sort of equation. Every time we integrate we must add in a constant of integration.

Therefore we expect there to be two arbitrary constants in our general solution to the equation. Indeed, in many physical systems we find that once we have described how the object moves under the action of the forces we are still free to choose a starting position and velocity for the object.

In this dissertation, we discuss the nonoscillatory property of the solutions of the second order liner differential equation

$$[r(t)u'(t)]' + c(t)u(t) = 0 \qquad (1.1)$$

and the second order half-liner differential equation

$$\{r(t)\phi[u'(t)]\}' + c(t)\phi[u(t)] = 0 \qquad (1.2)$$

where

(i) $r, c \in C([t_0, \infty), R := (-\infty, \infty))$ and $r(t) > 0$ on $[t_0, \infty)$ for some $t_0 \geq 0$;

(ii) $\phi(u) = |u|^{p-2} u$ for some fixed number $p > 1$.

Clearly, if $p = 2$ then (1.2) reduces to (1.1). By a solution of (1.2) will be meant a real-valued function $u(t)$ which is not identically zero on $[t_0, \infty)$ and satisfies (1.2)

Equations (1.1) or (1.2) is said to be nonoscillatory in $[t_0, \infty)$ if no solution of equations (1.1) or (1.2) vanishes more than once in this interval. The equation (1.1) or (1.2) will be said to be oscillatory if one (and therefore all) of its solutions have an infinite number of zeros on $[t_0, \infty)$.

Our main concern will be to obtain nonoscillatory (or oscillatory) criteria for equation (1.1) or (1.2), that is, conditions on the functions $r(t)$, $c(t)$ and ϕ from which conclusions may be drawn as to the nonoscillatory (or oscillatory) character of equation (1.1) or (1.2). There exists an entersive literature on this subject, see, for example, [1-14].

In [8], Li and Yeh obtained some nonoscillatory criteria of the second order differential equation (1.1) by using the subsituation $w(t) = \frac{u(t)}{\sqrt{a(t)}}$. In this note, we first will use another method for equation (1.1).

Using this result, we improve some results in [3,4,8,10]. In the third chapter, we extend a Leighton oscillatory criteria from equation (1.1) to the second order half-liner differential equation (1.2).

PRELIMINARIES

Definition:

A mapping *A* of a vector space X into a vector space *Y* is called a ***linear operator*** if

$$A(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 A x_1 + \alpha_2 A x_2$$

for all x_1, x_2 in X and all real α_1, α_2.

Definition:

A measurable function defined on [0, 1] is said to belong to **the *space***

$L^p = L^p[0,1]$ if $\int_0^1 |f|^p < \infty$.

Definition:

Let *C*(*t*) be a strongly continuous cosine family *C*(*t*), *t* ∈ *R*. We define a ***linear bounded operator*** $\Phi : L^2(0,T;X) \to X$ by

$$\Phi p = \int_0^T S(T-t)p(t)dt \qquad ,\text{for } p(\cdot) \in L^2(0,T;X)$$

where $S(t)$ is the associated sine family of $C(t)$.

Definition:

A complete normed linear space is called a **Banach space**.

Definition:

The nonempty subset $K_T(f)$ in X consisting of all terminal states of (1.1) at time T is called the **reachable set** at T of the system (1.1) starting at 0.

$$K_T(x_0, y_0) = \{x(T; x_0, y_0; 0, u) : u \in L^2(0,T;U)\}$$
$$\tilde{K}_T(x_0, y_0) = \{x(T; x_0, y_0; f, u) : u \in L^2(0,T;U)\}$$

Definition:

Let X be a Banach space. A one parameter family $T(t)$, $0 \le t < \infty$ of bounded linear operators from X into X is a **semigroup** of bounded linear operators on X if

(i) $T(0) = I$, (I is the identity operator on X.)

(ii) $T(t + s) = T(t)\, T(s)$ for every $t, s \ge 0$.

Definition:

(1) A semigroup $T(t)$, $0 \le t < \infty$ of bounded linear operators on X is a strongly continuous semigroup of bounded linear operators if

(2) $\lim\limits_{t \downarrow 0} T(t)x = x$, for every $x \in X$.

(3) A strongly continuous semigroup of bounded linear operators on X will be called a semigroup of class or simply a C_0 **semi group**.

STRONGLY CONTINUOUS COSINE FAMILY OF A DIFFERENTIAL EQUATION

Definition:

A one parameter family $C(t)$, t $\in$ R, of bounded linear operators mapping the Banach space X into itself is called a ***Strongly continuous cosine family*** if

(C_1) $C(s+t) + C(s-t) = 2C(s)C(t)$ for all $s, t \in R$

(C_2) $C(0) = I$

(C_3) $C(t)x$ is Continuous in t on R for each fixed $x \in X$.

If $C(t)$, $t \in$ R, is a strongly continuous cosine family in X, then $S(t)$, $t \in R$, is the one parameter family of operators in X defined by

$$S(t)x = \int_o^t C(s)x \; ds,\; x \in X, t \in R$$

Definition:

The ***infinitesimal generator of a strongly continuous cosine family*** $C(t)$, $t \in R$, is the operator $A : X \to X$ defined by

$$Ax = \frac{d^2}{dt^2} C(t)x \Big|_{t=0}$$

where

$D(A) = \{x \in X / C(t)x$ is a twice continuously differentiable function of $t\}$

E = $\{x \in X / C(t)x$ is a once continuously differentiable function of $t\}$.

Lemma:2.1

If $C(t)$, $t \in R$, is a strongly continuous cosine family in X, then

(i) There exist constants $k \geq 1$ and $w \geq 0$ so that

$$|C(t)| \leq k e^{w|t|} \quad for\ all\ t \in R \quad and$$

$$|S(t_1) - S(t_2)| \leq k\left|\int_{t_1}^{t_2} e^{w|s|} ds\right| \quad for\ all\ t_1, t_2 \in R$$

(ii) If $x \in E$, then $S(t)\, x \in D(A)$ and

$$(d/dt)C(t)x = AS(t)x.$$

Proof:

(i) by definition strongly continuous cosine family of a equation

$$S(t)x = \int_0^t C(s)x\, ds, \qquad x \in X, t \in R$$

$$\therefore \quad S(t_1)x = \int_0^{t_1} C(s)x\, ds, \qquad x \in X, t_1 \in R \qquad (1)$$

$$S(t_2)x = \int_0^{t_2} C(s)x\, ds, \qquad x \in X, t_2 \in R \qquad (2)$$

From equations (1) and (2) we have

$$[S(t_1)x - S(t_2)x] = \int_0^{t_1} C(s)x\, ds - \int_0^{t_2} C(s)x\, ds$$

$$= -\int_{t_1}^{0} C(s)x\, ds - \int_0^{t_2} C(s)x\, ds$$

$$[S(t_1) - S(t_2)]x = -\int_{t_1}^{t_2} C(s)x\, ds$$

$$|[S(t_1) - S(t_2)]x| = \left|-\int_{t_1}^{t_2} C(s)x\, ds\right|$$

$$|S(t_1) - S(t_2)||x| \leq |x|\left|\int_{t_1}^{t_2} C(s)x\, ds\right|$$

$$|S(t_1) - S(t_2)| \leq \left|\int_{t_1}^{t_2} C(s)x\, ds\right|$$

Given that

$$|C(t)| \leq k e^{w|t|}, \quad \text{for all } t \in R$$

$$\therefore \quad |S(t_1) - S(t_2)| \leq k\left|\int_{t_1}^{t_2} e^{w|s|}\, ds\right|$$

(ii) Given that $x \in E$

$\Rightarrow C(t)\, x$ is a once continuously differentiable function of t.

$$\Rightarrow \frac{d}{dt} C(t)x \text{ exists finitely}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{d}{dt} C(t)x\right) \text{exists finitely}$$

$\Rightarrow \dfrac{d^2}{dt^2} C(t)x$ exists

$\Rightarrow C(t)\,x$ is a twice continuously differentiable function of t.

$\therefore S(t)x = \int_o^t C(s)x\,ds$ is also twice continuously differentiable function of t.

$\Rightarrow S(t)\,x \in D(A)$

Therefore, if $x \in E$ then $S(t)\,x \in D(A)$.

Next we have to prove $\left(\dfrac{d}{dt}\right) C(t)x = AS(t)x$

By definition of *infinitesimal generator of a strongly continuous cosine family* $C(t)$,,

$$Ax = \frac{d^2}{dt^2} C(t)x \Big|_{t=0}$$

$$Ax = x\frac{d^2}{dt^2} C(t) \Big|_{t=0}$$

$$A = \frac{d^2}{dt^2} C(t) \Big|_{t=0}$$

$$= \frac{d}{dt}(-S(t)) \Big|_{t=0}$$

$$= -C(t)\Big|_{t=0}$$

$$= -C(0)$$

$$= -I$$

$$A = -1 \qquad (3)$$

$$\frac{d}{dt} C(t)x = -S(t)x$$

$$= (-1)S(t)x$$

$$\therefore \quad \frac{d}{dt} C(t)x = AS(t)x$$

Hence the proof

Remark: 1

Let $C(t),\ t \in R$, be a strongly continuous cosine family in X with infinitesimal generator A. If $g : R \rightarrow X$ is continuously differentiable, $x_1 \in D(A)$, $x_2 \in E$ and

$$x(t) = C(t)x_1 + S(t)x_2 + \int_0^t S(t-s)g(s)ds, \quad t \in R.$$

then $x(t) \in D(A)$ for $t \in R$, x is twice continuously differentiable, and x satisfies

$$x''(t) = Ax(t) + g(t), \qquad x(0) = x_1, \qquad x'(0) = x_2 \ .$$

Remark : 2

Let $T(t)$ be a C_0 semigroup and let A be its infinitesimal generator. Then prove that

(a) For $x \in X$, $\lim_{h\to 0}\frac{1}{h}\int_t^{t+h} T(s)xds = T(t)x$

(b) For $x \in X$, $\int_0^t T(s)xds \in D(A)$ and $A\left(\int_0^t T(s)xds\right) = T(t)x - x$

Note:

We consider the following mild solution of equation (1.1)

$$x(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)\int_0^s f(s,\tau,x(\tau))d\tau ds$$
$$+\int_0^t S(t-s)Bu(s)ds$$

OSCILLATION CRITERIA FOR EQUATION

Let $u(t)$ be a solution of (1.1). According to the Kummer transformation (see Kwong and Zettl [5] or Willett [14]), we define

$$w(t) = \frac{u(t)}{\sqrt{a(t)}} \qquad \text{on} \qquad [t_0,\infty),$$

where $a(t) \in C^2([t_0,\infty),(0,\infty))$ is a given function. Then (1.1) is transformed into

$$(a(t)r(t)w'(t))' + \varphi(t)w(t) = 0 \qquad (1)$$

where $\varphi(t) := a(t)[c(t) + r(t)f^2(t) - (r(t)f(t))']$ and $f(t) := -\frac{a'(t)}{2a(t)}$.

Hence, equation (1.1), (1) and the following differential equation are equivalent:

$$\begin{aligned}&(a_1(t)a(t)r(t)v'(t))' + a_1(t)[\phi(t)\\&+a(t)r(t)g^2(t) - (a(t)r(t)g(t))']v(t) = 0\end{aligned} \qquad (2)$$

where $a_1(t) \in C^2([t_0,\infty),(0,\infty))$ and $g(t) = -\frac{a_1(t)}{2a_1(t)}$ on $[t_0,\infty)$.

Using these equivalent relations, Li and Yeh [8] established the following nonoscillatory characterization for equation (1.1) as follows:

Theorem: 3.1

Equation (1.1) is nonoscillatory if and only if one of the following conditions holds:

(a) there exists a function $f \in C([T,\infty),R)$ for some $T \geq t_0$ such that

$$c(t) + r(t)f^2(t) - (r(t)f(t))' \leq 0 \qquad \text{on} \qquad [t_0,\infty).$$

(b) there exists a function $v \in C^1([T,\infty),R)$ for some $T \geq t_0$ such that

$$c(t) + r(t)v^2(t) + \varphi(t) - (a(t)r(t)v(t))' \leq 0,\ t \geq T.$$

where $a(t) \in C^2([t_0,\infty),(0,\infty))$ is a given function and $\varphi(t) = a(t)[c(t) + r(t)f^2(t) - (r(t)f(t))']$.

Clearly, condition (b) is condition (a) if $a(t)=1$.

We also have the following observation:

If $c(t)\le 0$ for t large enough, then equation (1.1) is nonoscillatory. Suppose that "$c(t)\le 0$ for t large enough" does not hold. If we can find a, $a_1 \in C^2([t_0,\infty),(0,\infty))$ such that the coefficient of $w(t)$ and $v(t)$ in (1) or (2) is nonpositive, then equation (1.1) is nonoscillatory.

Using Theorem 3.1 Li and Yeh [8] obtained many nonoscillatory criteria for equation (1.1). In this chapter, we use another method to derive Theorem 3.1. Using this result, we establish some nonoscillatory criteria which generalize some results of Hille [3], Kneser [4] and Li-Yeh [8]. An alternative proof of the Sturm comparison theorem [10] is also given. For other related results, we refer to [2,6,10].

Throughout this chapter, we assumed that $a(t)\in C^2([t_0,\infty),(0,\infty))$ is a given function,

$$f(t):=-\frac{a'(t)}{2a(t)} \qquad \text{and}$$

$$\varphi(t):=a(t)[c(t)+r(t)f^2(t)-(r(t)f(t))']=a(t)(c(t)+\frac{v^2(t)}{r(t)}+v'(t)).$$

Hence

$$v(t):=-r(t)f(t).$$

Now, we can state and prove our main result as follows:

Leighton's oscillation

In 1950, Leighton [6] showed the following oscillation criterion:

Leighton's Oscillation Theorem.

If $\int^{\infty}\frac{1}{r(t)}dt=\int^{\infty}c(t)dt=\infty$, then equation (1.1) is nonoscillatory.

In this chapter, we will extend Leighton's Oscillation Theorem to the second order half-linear ordinary differential equation (1.2) by using the Coles' technique [1].

Theorem: 4.1

If $\int^{\infty}c(t)dt=\int^{\infty}r^{1-q}(t)dt=\infty$, where $\frac{1}{p}+\frac{1}{q}=1$, then equation (1.2) is oscillatory.

Proof.

Suppose not.

Then (1.2) has a nonoscillatory solution $u(t)\ne 0$ on $[T,\infty)$ for some $T\ge t_0$.

Define

$$v(t)=\frac{r(t)\phi(u'(t))}{\phi(u(t))}, \qquad t\ge T.$$

Then, for $t\ge T$,

$$v'(t)=-c(t)-\frac{r(t)\phi(u'(t))\phi'(u(t))}{\phi^2(u(t))}$$

$$=-c(t)-\frac{r(t)u'(t)\,|\,u'(t)\,|^{p-2}\,(p-1)u^{p-2}(t)u'(t)}{u^{2p-2}(t)}$$

$$=-c(t)-\frac{(p-1)r(t)\,|\,u'(t)\,|^{p}}{u^{p}(t)}$$

$$= -c(t) - (p-1) r^{1-\frac{p}{p-1}}(t) [\frac{r(t)|u'(t)|^{p-1}}{u^{p-1}(t)}]^{\frac{p}{p-1}}$$

$$= -c(t) - (p-1) r^{1-q}(t) | v(t) |^q$$

Thus, for $t \geq T$,

$$v'(t) + c(t) + (p-1) r^{1-q}(t) | v(t) |^q = 0. \qquad (1)$$

It follows from (1) that, for $t \geq T$,

$$v(t) = v(t_0) - \int_{t_0}^t c(s)ds - \int_{t_0}^t (p-1) r^{1-q}(s) | v(s) |^q \ ds.$$

Since $\int^{\infty} c(t)dt = \infty$, we can always find $t_1 \geq t_0$ such that

$$v(t_0) - \int_{t_0}^t c(s)ds < 0 \text{ for all } t \in [t_1, \infty).$$

Thus

$$v(t) < \int_{t_0}^t (p-1) r^{1-q}(s) | v(s) |^q \ ds \text{ for all } t \geq t_1.$$

Let

$$R(t) := \int_{t_0}^t (p-1) r^{1-q}(s) | v(s) |^q \ ds,$$

Then $R(t) > 0$, $| v(t) |^q > R^q(t)$ and

$$R'(t) = (p-1) r^{1-q}(t) | v(t) |^q > (p-1) r^{1-q}(t) R^q(t), \text{ for } t \geq t_1 \geq t_0$$

Thus,

$$\frac{R'(t)}{R^q(t)} > (p-1) r^{1-q}(t).$$

Integrating it from t_1 to t, we have

$$\frac{-R^{1-q}(t_1)}{1-q} > \frac{1}{1-q}(R^{1-q}(t) - R^{1-q}(t_1)) = \int_{t_1}^t \frac{dR(s)}{R^q(s)} > \int_{t_1}^t (p-1) r^{1-q}(s) R^q(s) ds$$

Letting $t \to \infty$,

$$\infty > \frac{-R^{1-q}(t_1)}{1-q} > (p-1) \int_{t_1}^{\infty} r^{1-q}(s) ds = \infty,$$

which is a contradiction.

Thus (1.2) is oscillatory.

Remark:1

Let $p = 2$. Then Theorem 4.1 reduces to Leighton's Oscillatory Theorem.

Using Leighton's Oscillatory Theorem, we have the following:

Corollory:1

Let a, $a_1 \in C^2([t_0, \infty), (0, \infty))$. If either

$$\int^{\infty} \frac{1}{a(t) r(t)} dt = \int^{\infty} \varphi(t) dt = \infty$$

or

$$\int^{\infty} \frac{1}{a_1(t) a(t) r_1(t)} dt = \int^{\infty} a(t) [\varphi(t) + a(t) r(t) g^2(t) - (a(t) r(t) g(t))'] dt = \infty$$

where $\varphi(t)$ and $g(t)$ are defined as in chapter 1, then equation (1.1) is oscillatory.

CONCLUSION

There is no cosine families for an invariant space. So, the limitations of these families are not authentic. We note that the same family considered as acting on the larger banach space of bounded, uniformly continuous functions vanishing at 0, belongs to C.

While this result and that of examples are similar in content. There is a difference between the arguments used for their establishment. Hence we gave the existence and uniqueness of solutions for second order differential equations. In this present research, the result is generalized by the cosine family of oscillation criterion is nonoscillatory.

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