SOLUTION OF THE PELL EQUATIONS

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ABSTRACT:

In this paper we discussed all positive integer solutions of the equation $x^2 - dy^2 = \pm 1$ and $x^2 - dy^2 = \pm 4$ are given in terms of Pell Fibonacci and Pell Lucas equation.

Key word: Diophantine equations, Pell equations, continued fraction, n[th] convergent, Quadratic irrational.

INTRODUCTION:

In mathematics, a Diophantine equation is a polynomial equation in two or more unknowns such that only the integer solutions are searched or studied. In Diophantine equation, $x^2 - dy^2 = \pm N$ is a Pell's equation[1]. Let $d \neq 1$ be a positive nonsquare integer and N- be any fixed positive integer[2]. Pell's equation is named after the English mathematician John Pell[3]. It was studied by Brahmagupta in the 7[th] century as well as by Fermat in the 17[th] century. The Pell equation has infinitely many integer solutions $(x_n\,,\ y_n)$ for $n \geq 1$ [3]. The first nontrivial positive integer solution $(x_1\,,\ y_1)$ of this equation is called the fundamental solution,

The aim of this Paper is to study an application of continued fraction for some kind of Pell's equation and the salient aspects of the subject matter in four sections.

Section 1 and Section 2 contains preliminaries and basic theorem of solutions of the Pell equations $x^2 - dy^2 = N$ [4].

Section 3 deals with to find the fundamental solution of the equation $x^2 - (k^2 + k)y^2 = 2^t$ and generalized the solution.

Section 4 deals with to find the fundamental solution of the equation $x^2 - (a^2b^2 + 2b)y^2 = N$, when $N \in \{\pm 1, \pm 4\}$, and generalized the solution through Fibonacci and Lucas sequences.

RELIMINARIES and BASIC DEFINITIONS

Definition 2.1 Let $a_0,\ a_1,\ \dots,\ a_m -$ be real numbers. Then

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots \cfrac{1}{a_{m-1} + \cfrac{1}{a_m}}}}$$

is called a **finite continued fraction**[8] and is denoted by $[a_0,\ a_1,\ \dots,\ a_m]$.

Definition 2.2 If the chain of fractions does not stop, then it is called an **infinite continued fraction**[8].

Definition 2.3 (a) For $n \leq m$, $[a_0,\ a_1,\ \dots,\ a_n] -$ is called $\boldsymbol{n^{th}}$ **convergent**[7] to $[a_0,\ a_1,\ \dots,\ a_m]$. (b) Define two sequences of real numbers, (p_n) and (q_n), recursive as follows,(i) $p_{-1} = 1$, $p_0 = a_0$ and $p_n = a_n\, p_{n-1} + p_{n-2}$ (ii) $q_{-1} = 0$, $q_0 = 1$, and $q_n = a_n\, q_{n-1} + q_{n-2}$.

Definition 2.4 Let $\alpha -$ be a real number. For $n = 0,1,2,\ldots$ define a recursive algorithm as follows, $\alpha_0 = \alpha$, $a_n = |\alpha_n|$ and $\alpha_n = a_n + \frac{1}{\alpha_{n+1}}$, where, α_n, $a_n \geq 1$, for $n \geq 1$.

Given positive m, $\alpha = [a_0, \ a_1, \ \ldots, a_{m-1}, \ a_m]$. It is called **$\mathbf{m^{th}}$ continued fraction** [7] of α .

Definition 2.5 Let $\alpha -$ be an irrational number. $\alpha -$ is called a **quadratic irrational** if it is a root of integer polynomial of degree two.

The other root $\beta -$ is called a conjugate of α .

Definition 2.6 Let $[a_0, \ a_1, \ \ldots, a_n, \ldots] -$ be a continued fraction such that $a_n = a_{n+l}$ for all sufficiently large n and a fixed positive integer l. Then it is **periodic** and $l -$ is called a period.

SOLUTIONS OF THE PELL EQUATIONS

$$\boldsymbol{x^2 - (k^2 + k)y^2 = N, \quad When \quad N \in 2^t}$$

In this work we will define by recurrence an infinite sequence of positive integer solutions of the Pell equation $x^2 - dy^2 = 2^t$, where $d = k^2 + k$ with $k \geq 1$ an integer and $t \geq 0$ is also an integer. First we consider the case $t = 0$.

That is, the classical Pell equation, $x^2 - (k^2 + k)y^2 = 1$.

MAIN RESULTS

Theorem 3.1 Let $d = k^2 + k$ with $k \geq 1$. Then,

(1) The continued fraction expansion of $\sqrt{d}$ is,

$$\sqrt{d} = \begin{cases} [1\,;\overline{2}] & \text{if } k = 1, \\ [k\,;\overline{2, 2k}\,] & \text{otherwise} \end{cases}$$

(2) The fundamental solution of $x^2 - dy^2 = 1$ is $(x_1, y_1) = (2k + 1, 2)$.

(3) For $n \geq 4$,

$$x_n = (4k + 3)(x_{n-1} - x_{n-2}) + x_{n-3}$$
$$y_n = (4k + 3)(y_{n-1} - y_{n-2}) + y_{n-3}$$

Proof.

Let $k = 1$, $\quad \sqrt{k^2 + k} = \sqrt{1 + 1} = \sqrt{2}$

$$x^2 - 2y^2 = 1.$$

The simple continued fraction of $\sqrt{2}$ is

$$\sqrt{2} = [1\,;\ \overline{2}].$$

Now, let $k \geq 2$,

$$\begin{aligned} \sqrt{k^2 + k} &= k + \sqrt{k^2 + k} - k \\ &= k + \cfrac{1}{\cfrac{1}{\sqrt{k^2 + k} - k}} \\ &= k + \cfrac{1}{\cfrac{\sqrt{k^2 + k} + k}{k^2 + k - k^2}} \\ &= k + \cfrac{1}{\cfrac{\sqrt{k^2 + k} + k}{k}} \\ &= k + \cfrac{1}{\cfrac{\sqrt{k^2 + k} + 2k - k}{k}} \end{aligned}$$

$$= k + \cfrac{1}{2 + \cfrac{\sqrt{k^2+k} - k}{k}}$$

$$= k + \cfrac{1}{2 + \cfrac{1}{\cfrac{k}{\sqrt{k^2+k} - k}}}$$

$$= k + \cfrac{1}{2 + \cfrac{1}{\cfrac{k\left(\sqrt{k^2+k} + k\right)}{k^2 + k - k^2}}}$$

$$= k + \cfrac{1}{2 + \cfrac{1}{\sqrt{k^2+k} + 2k - k}}$$

$$= k + \cfrac{1}{2 + \cfrac{1}{2k + \cfrac{1}{\cfrac{1}{\sqrt{k^2+k} - k}}}}$$

$$= k + \cfrac{1}{2 + \cfrac{1}{2k + \cfrac{1}{2 + \cfrac{1}{2k + \cfrac{1}{\cfrac{1}{\sqrt{k^2+k} - k}}}}}}$$

Therefore, the continued fraction expansion of $\sqrt{k^2+k}$ – is $\sqrt{k^2+k} = \left[k\,;\,\overline{2\,,2k}\right]$.

(2) The case $k = 1$ is clear.

Since, $(x_1, y_1) = (3\,,2)$ is minimum solution of $x^2 - 2y^2 = 1$.

$k \geq 2 \;\Rightarrow\; a_0 = k, \qquad a_1 = 2$

By the definition (2.2.3)

$$\begin{aligned}(x_1, y_1) &= (p_1, q_1)\\ &= (1 + a_0 a_1\,, a_1)\\ &= (1 + 2k, 2)\end{aligned}$$

Therefore, the fundamental solution of the equation

$$x^2 - 2y^2 = 1 \text{ is } (1 + 2k\,,2).$$

(3) For $n \geq 4$.

The fundamental solution of $x^2 - (k^2 + k)y^2 = 1$ is $(x_1\,, y_1)$.

The other solutions of the equation are $(x_n\,, y_n)$, which is derived by using the equalities

$$\left(x_n + \sqrt{d}y_n\right) = \left(x_n + \sqrt{d}y_n\right)^n, \qquad \text{for} \quad n \geq 2.$$

In other words,

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 & dy_1 \\ y_1 & x_1 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \text{for} \quad n \geq 2.$$

Therefore it can be shown by induction on n that,

$$x_n = (4k + 3)(x_{n-1} - x_{n-2}) + x_{n-3} \quad \text{and}$$

$$y_n = (4k + 3)(y_{n-1} - y_{n-2}) + y_{n-3}\,, \quad \text{for} \quad n \geq 4.$$

This proof is complete.

Now, the general case, $x^2 - (k^2 + k)y^2 = 2^t$, for $t \geq 1$, we have consider the two cases.

Let $k = 1$ and $k \geq 2$.

(x_n , y_n) – be the integer solutions of $x^2 - (k^2 + k)y^2 = 1$ and
(u_n , v_n) – be the integer solutions of $x^2 - (k^2 + k)y^2 = 2^t$.

SOLUTIONS OF THE PELL EQUATIONS

$$\boldsymbol{x^2 - (a^2b^2 + 2b)y^2 = N, \quad When \quad N \in \{\pm 1 , \pm 4\}}$$

In this chapter, we get all positive integer solutions of
$x^2 - (a^2b^2 + 2b)y^2 = \pm 1$ and $x^2 - (a^2b^2 + 2b)y^2 = \pm 4$ are given interms of the generalized Fibonacci and Lucas sequences.

MAIN RESULTS

Theorem 4.1 Let $d = a^2b^2 + 2b$. Then $\sqrt{d} = [ab , \overline{a , 2ab}]$.

Proof.

$$\sqrt{d} = \sqrt{a^2b^2 + 2b}$$
$$= ab + \sqrt{a^2b^2 + 2b} - ab$$
$$= ab + \cfrac{1}{\cfrac{1}{\sqrt{a^2b^2 + 2b} - ab}}$$
$$= ab + \cfrac{1}{\cfrac{\sqrt{a^2b^2 + 2b} + ab}{2b}}$$
$$= ab + \cfrac{1}{\cfrac{2ab}{2b} + \cfrac{\sqrt{a^2b^2 + 2b} - ab}{2b}}$$
$$= ab + \cfrac{1}{a + \cfrac{1}{\cfrac{2b}{\sqrt{a^2b^2 + 2b} - ab}}}$$
$$= ab + \cfrac{1}{a + \cfrac{1}{2ab + \cfrac{1}{a + \cfrac{1}{2ab} \cdots}}}$$
$$\sqrt{d} = \sqrt{a^2b^2 + 2b} = [ab , \overline{a , 2ab}].$$

This proof is complete.

Theorem 4.2.2.

Let $d = a^2b^2 + 2b$. If $b \neq 1$, then $\sqrt{d} = [ab , \overline{2a , 2ab}]$ and if $b = 1$ then $\sqrt{d} = [a , \overline{2a}]$.

Proof.

$$\sqrt{d} = \sqrt{a^2b^2 + b}$$
$$= ab + \sqrt{a^2b^2 + b} - ab$$

$$= ab + \cfrac{1}{\cfrac{1}{\sqrt{a^2b^2+b} - ab}}$$

$$= ab + \cfrac{1}{2a + \cfrac{1}{2ab + \cfrac{1}{2a + \cfrac{1}{2ab} \cdots}}}$$

$$\sqrt{d} = \sqrt{a^2b^2+b} = \left[ab\,,\overline{2a\,,2ab}\right].$$

If $b = 1$, then,

$$\sqrt{d} = [a\,,\overline{2a}].$$

This proof is complete.

CONCLUSION:

In this Paper, by using continued fraction expansion of $\sqrt{d}$, we find fundamental solution of the $x^2 - dy^2 = \pm 1$, where a, b and c are natural numbers and $d = a^2b^2c^2 + 2ab$ or $a^2b^2c^2 - 2ab$. Moreover, we investigate Pell equation of the form $x^2 - dy^2 = N$ when $N = \pm 1, \pm 4$ and we are looking for positive integer solutions in x and y. We get all positive integer solutions of the Pell equations $x^2 - dy^2 = N$ in terms of generalized Fibonacci and Lucas sequences when $N = \pm 1, \pm 4$ and $a^2b^2c^2 + 2ab$, $a^2b^2c^2 - 2ab$. Finally, all positive integer solutions of the equations $x^2 - dy^2 = \pm 1$ and $x^2 - dy^2 = \pm 4$ are given in terms of Pell - Fibonacci and Pell-Lucas sequences.

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